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LIST OF CONJECTURAL SERIES FOR POWERS OF π AND OTHER CONSTANTS

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ABSTRACT. Here I give the full list of my conjectures on series for powers of π and other important constants scattered in some of my public preprints or my private diaries. The list contains totally 170 conjectural series. On the list there are 162 series for π^{-1} , three series for π^2 , two series for π^{-2} , one series for $\zeta(3)$, and two series for $L(2, (\frac{\cdot}{3}))$. The code of a conjectural series is underlined if and only if a complete proof of the identity is available now.

1. VARIOUS SERIES FOR SOME CONSTANTS

Conjecture 1. (i) (Z. W. Sun [S11, Conj. 1.4]) *We have*

$$\sum_{k=1}^{\infty} \frac{(10k-3)8^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = \frac{\pi^2}{2}, \quad (1.1)$$

$$\sum_{k=1}^{\infty} \frac{(11k-3)64^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = 8\pi^2, \quad (\underline{1.2})$$

$$\sum_{k=1}^{\infty} \frac{(35k-8)81^k}{k^3 \binom{2k}{k}^2 \binom{4k}{2k}} = 12\pi^2, \quad (1.3)$$

$$\sum_{k=1}^{\infty} \frac{(15k-4)(-27)^{k-1}}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = K, \quad (\underline{1.4})$$

$$\sum_{k=1}^{\infty} \frac{(5k-1)(-144)^k}{k^3 \binom{2k}{k}^2 \binom{4k}{2k}} = -\frac{45}{2}K, \quad (1.5)$$

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where

$$\begin{aligned} K &:= L\left(2, \left(\frac{\cdot}{3}\right)\right) = \sum_{k=1}^{\infty} \frac{\left(\frac{k}{3}\right)}{k^2} \\ &= 0.781302412896486296867187429624\dots \end{aligned}$$

with $(-)$ the Legendre symbol.

(ii) (Z. W. Sun [S, Conj. A40]) We have

$$\sum_{k=1}^{\infty} \frac{(28k^2 - 18k + 3)(-64)^k}{k^5 \binom{2k}{k}^4 \binom{3k}{k}} = -14\zeta(3), \quad (1.6)$$

where

$$\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

(iii) (Z. W. Sun [S11, Conj. 1.4]) We have

$$\sum_{n=0}^{\infty} \frac{18n^2 + 7n + 1}{(-128)^n} \binom{2n}{n}^2 \sum_{k=0}^n \binom{-1/4}{k}^2 \binom{-3/4}{n-k}^2 = \frac{4\sqrt{2}}{\pi^2} \quad (1.7)$$

and

$$\sum_{n=0}^{\infty} \frac{40n^2 + 26n + 5}{(-256)^n} \binom{2n}{n}^2 \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} = \frac{24}{\pi^2}. \quad (1.8)$$

Remark. (a) I announced (1.1)-(1.6) first by several messages to **Number Theory Mailing List** during March-April in 2010. My conjectural identity (1.2) was confirmed by J. Guillera [G] in December 2010 via applying a Barnes-integrals strategy of the WZ-method. In 2011 K. Hessami Pilehrood and T. Hessami Pilehrood [HP] proved my conjectural identity (1.4) by means of the Hurwitz zeta function.

(b) L. van Hamme [vH] investigated corresponding p -adic congruences for certain hypergeometric series involving the Gamma function or $\pi = \Gamma(1/2)^2$. All of my conjectural series were motivated by their p -adic analogues that I found first. For example, I discovered (1.6) soon after I found the conjectural congruence

$$\sum_{k=0}^{p-1} \frac{28k^2 + 18k + 3}{(-64)^k} \binom{2k}{k}^4 \binom{3k}{k} \equiv 3p^2 - \frac{7}{2}p^5 B_{p-3} \pmod{p^6}$$

for any odd prime p , where B_0, B_1, B_2, \dots are Bernoulli numbers. The reader may consult [S], [S11], [S11a], [S11b] and [S11c] for many congruences related to my conjectural series.

Conjecture 2 (i) ([S11b]). *Set*

$$a_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} x^{n-k} \quad (n = 0, 1, 2, \dots).$$

Then we have

$$\sum_{k=0}^{\infty} \frac{13k+4}{96^k} \binom{2k}{k} a_k(-8) = \frac{9\sqrt{2}}{2\pi}, \quad (2.1)$$

$$\sum_{k=0}^{\infty} \frac{290k+61}{1152^k} \binom{2k}{k} a_k(-32) = \frac{99\sqrt{2}}{\pi}, \quad (2.2)$$

$$\sum_{k=0}^{\infty} \frac{962k+137}{3840^k} \binom{2k}{k} a_k(64) = \frac{252\sqrt{5}}{\pi}. \quad (2.3)$$

(ii) ((2.4)-(2.23) and (2.24)-(2.28) were discovered during March 27-31, 2011 and Jan. 23-24, 2012 respectively) *For $n = 0, 1, 2, \dots$ define*

$$S_n^{(1)}(x) = \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} x^{n-k}$$

and

$$S_n^{(2)}(x) = \sum_{k=0}^n \binom{2k}{k}^2 \binom{2n-2k}{n-k} x^{n-k}.$$

Then we have

$$\sum_{k=0}^{\infty} \frac{12k+1}{400^k} \binom{2k}{k} S_k^{(1)}(16) = \frac{25}{\pi}, \quad (2.4)$$

$$\sum_{k=0}^{\infty} \frac{10k+1}{(-384)^k} \binom{2k}{k} S_k^{(1)}(-16) = \frac{8\sqrt{6}}{\pi}, \quad (2.5)$$

$$\sum_{k=0}^{\infty} \frac{170k+37}{(-3584)^k} \binom{2k}{k} S_k^{(1)}(64) = \frac{64\sqrt{14}}{3\pi}, \quad (2.6)$$

$$\sum_{k=0}^{\infty} \frac{476k+103}{3600^k} \binom{2k}{k} S_k^{(1)}(-64) = \frac{225}{\pi}, \quad (2.7)$$

$$\sum_{k=0}^{\infty} \frac{140k+19}{4624^k} \binom{2k}{k} S_k^{(1)}(64) = \frac{289}{3\pi}, \quad (2.8)$$

$$\sum_{k=0}^{\infty} \frac{1190k+163}{(-4608)^k} \binom{2k}{k} S_k^{(1)}(-64) = \frac{576\sqrt{2}}{\pi}. \quad (2.9)$$

Also,

$$\sum_{k=0}^{\infty} \frac{k-1}{72^k} \binom{2k}{k} S_k^{(2)}(4) = \frac{9}{\pi}, \quad (2.10)$$

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-192)^k} \binom{2k}{k} S_k^{(2)}(4) = \frac{\sqrt{3}}{\pi}, \quad (2.11)$$

$$\sum_{k=0}^{\infty} \frac{k-2}{100^k} \binom{2k}{k} S_k^{(2)}(6) = \frac{50}{3\pi}, \quad (2.12)$$

$$\sum_{k=0}^{\infty} \frac{k}{(-192)^k} \binom{2k}{k} S_k^{(2)}(-8) = \frac{3}{2\pi}, \quad (2.13)$$

$$\sum_{k=0}^{\infty} \frac{6k-1}{256^k} \binom{2k}{k} S_k^{(2)}(12) = \frac{8\sqrt{3}}{\pi}, \quad (2.14)$$

$$\sum_{k=0}^{\infty} \frac{17k-224}{(-225)^k} \binom{2k}{k} S_k^{(2)}(-14) = \frac{1800}{\pi}, \quad (2.15)$$

$$\sum_{k=0}^{\infty} \frac{15k-256}{289^k} \binom{2k}{k} S_k^{(2)}(18) = \frac{2312}{\pi}, \quad (2.16)$$

$$\sum_{k=0}^{\infty} \frac{20k-11}{(-576)^k} \binom{2k}{k} S_k^{(2)}(-32) = \frac{90}{\pi}, \quad (2.17)$$

$$\sum_{k=0}^{\infty} \frac{10k+1}{(-1536)^k} \binom{2k}{k} S_k^{(2)}(-32) = \frac{3\sqrt{6}}{\pi}, \quad (2.18)$$

$$\sum_{k=0}^{\infty} \frac{3k-2}{640^k} \binom{2k}{k} S_k^{(2)}(36) = \frac{5\sqrt{10}}{\pi}, \quad (2.19)$$

$$\sum_{k=0}^{\infty} \frac{12k+1}{1600^k} \binom{2k}{k} S_k^{(2)}(36) = \frac{75}{8\pi}, \quad (2.20)$$

$$\sum_{k=0}^{\infty} \frac{24k+5}{3136^k} \binom{2k}{k} S_k^{(2)}(-60) = \frac{49\sqrt{3}}{8\pi}, \quad (2.21)$$

$$\sum_{k=0}^{\infty} \frac{14k+3}{(-3072)^k} \binom{2k}{k} S_k^{(2)}(64) = \frac{6}{\pi}, \quad (2.22)$$

$$\sum_{k=0}^{\infty} \frac{20k-67}{(-3136)^k} \binom{2k}{k} S_k^{(2)}(-192) = \frac{490}{\pi}, \quad (2.23)$$

$$\sum_{k=0}^{\infty} \frac{7k-24}{3200^k} \binom{2k}{k} S_k^{(2)}(196) = \frac{125\sqrt{2}}{\pi}, \quad (2.24)$$

and

$$\sum_{k=0}^{\infty} \frac{5k-32}{(-6336)^k} \binom{2k}{k} S_k^{(2)}(-392) = \frac{495}{2\pi}, \quad (2.25)$$

$$\sum_{k=0}^{\infty} \frac{66k-427}{6400^k} \binom{2k}{k} S_k^{(2)}(396) = \frac{1000\sqrt{11}}{\pi}, \quad (2.26)$$

$$\sum_{k=0}^{\infty} \frac{34k-7}{(-18432)^k} \binom{2k}{k} S_k^{(2)}(-896) = \frac{54\sqrt{2}}{\pi}, \quad (2.27)$$

$$\sum_{k=0}^{\infty} \frac{24k-5}{18496^k} \binom{2k}{k} S_k^{(2)}(900) = \frac{867}{16\pi}. \quad (2.28)$$

Remark. (i) Those $a_n(1)$ ($n = 0, 1, 2, \dots$) were first introduced by R. Apéry in his study of the irrationality of $\zeta(2)$ and $\zeta(3)$. Identities related to the form $\sum_{k=0}^{\infty} (bk+c) \binom{2k}{k} a_k(1)/m^k = C/\pi$ were first studied by T. Sato in 2002.

(ii) I introduced the polynomials $S_n^{(1)}(x)$ and $S_n^{(2)}(x)$ during March 27-28, 2011. By **Mathematica**, we have

$$S_n^{(1)}(-1) = \begin{cases} \binom{n}{n/2}^2 & \text{if } 2 \mid n, \\ 0 & \text{if } 2 \nmid n. \end{cases}$$

I also noted that

$$S_n^{(1)}(1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}^2 4^{n-2k}.$$

Identities of the form

$$\sum_{n=0}^{\infty} \frac{bn+c}{m^n} \binom{2n}{n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}^2 4^{n-2k} = \frac{C}{\pi}$$

were recently investigated in [CC].

(iii) In [S11c] I proved the following three identities via Ramanujan-type series for $1/\pi$ (cf. [B, pp. 353-354]).

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{k}{128^k} \binom{2k}{k} S_k^{(2)}(4) &= \frac{\sqrt{2}}{\pi}, \\ \sum_{k=0}^{\infty} \frac{8k+1}{576^k} \binom{2k}{k} S_k^{(2)}(4) &= \frac{9}{2\pi}, \\ \sum_{k=0}^{\infty} \frac{8k+1}{(-4032)^k} \binom{2k}{k} S_k^{(2)}(4) &= \frac{9\sqrt{7}}{8\pi}. \end{aligned}$$

In December 2011 A. Meurman [M] confirmed my conjectural (2.10).

Conjecture 3. (i) (Discovered on April 1, 2011) Set

$$W_n(x) := \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k}^2 \binom{2n-2k}{n-k} x^{-(n+k)} \quad (n = 0, 1, 2, \dots).$$

Then

$$\sum_{k=0}^{\infty} (8k+3)W_k(-8) = \frac{28\sqrt{3}}{9\pi}, \quad (3.1)$$

$$\sum_{k=0}^{\infty} (8k+1)W_k(12) = \frac{26\sqrt{3}}{3\pi}, \quad (3.2)$$

$$\sum_{k=0}^{\infty} (24k+7)W_k(-16) = \frac{8\sqrt{3}}{\pi}, \quad (3.3)$$

$$\sum_{k=0}^{\infty} (360k+51)W_k(20) = \frac{210\sqrt{3}}{\pi}, \quad (3.4)$$

$$\sum_{k=0}^{\infty} (21k+5)W_k(-28) = \frac{63\sqrt{2}}{8\pi}, \quad (3.5)$$

$$\sum_{k=0}^{\infty} (7k+1)W_k(32) = \frac{11\sqrt{2}}{3\pi}, \quad (3.6)$$

$$\sum_{k=0}^{\infty} (195k+31)W_k(-100) = \frac{275\sqrt{6}}{8\pi}, \quad (3.7)$$

$$\sum_{k=0}^{\infty} (39k+5)W_k(104) = \frac{91\sqrt{6}}{12\pi}, \quad (3.8)$$

$$\sum_{k=0}^{\infty} (2856k+383)W_k(-196) = \frac{637\sqrt{3}}{\pi}, \quad (3.9)$$

$$\sum_{k=0}^{\infty} (14280k+1681)W_k(200) = \frac{3350\sqrt{3}}{\pi}. \quad (3.10)$$

(ii) (Discovered during April 7-10, 2011; (3.17)-(3.19) appeared in [S11b])
Define

$$f_n^+(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} x^{2k-n}$$

and

$$f_n^-(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} (-1)^k x^{2k-n}$$

for $n = 0, 1, 2, \dots$. Then

$$\sum_{k=0}^{\infty} \frac{19k+3}{240^k} \binom{2k}{k} f_k^+(6) = \frac{35\sqrt{6}}{4\pi}, \quad (3.11)$$

$$\sum_{k=0}^{\infty} \frac{135k+8}{289^k} \binom{2k}{k} f_k^+(14) = \frac{6647}{14\pi}, \quad (3.12)$$

$$\sum_{k=0}^{\infty} \frac{297k+41}{2800^k} \binom{2k}{k} f_k^+(14) = \frac{325\sqrt{14}}{8\pi}, \quad (3.13)$$

$$\sum_{k=0}^{\infty} \frac{770k+79}{576^k} \binom{2k}{k} f_k^+(21) = \frac{468\sqrt{7}}{\pi}, \quad (3.14)$$

$$\sum_{k=0}^{\infty} \frac{209627k+22921}{46800^k} \binom{2k}{k} f_k^+(36) = \frac{58275\sqrt{26}}{4\pi}, \quad (3.15)$$

$$\sum_{k=0}^{\infty} \frac{205868k+18903}{439280^k} \binom{2k}{k} f_k^+(76) = \frac{1112650\sqrt{19}}{81\pi}, \quad (3.16)$$

$$\sum_{k=0}^{\infty} \frac{8851815k+1356374}{(-29584)^k} \binom{2k}{k} f_k^+(175) = \frac{1349770\sqrt{7}}{\pi}, \quad (3.17)$$

and

$$\sum_{k=0}^{\infty} \frac{1054k+233}{480^k} \binom{2k}{k} f_k^-(8) = \frac{520}{\pi}, \quad (3.18)$$

$$\sum_{k=0}^{\infty} \frac{224434k+32849}{5760^k} \binom{2k}{k} f_k^-(18) = \frac{93600}{\pi}. \quad (3.19)$$

(iii) ([S11b]) Define $g_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} x^k$ for $n = 0, 1, 2, \dots$. Then

$$\sum_{k=0}^{\infty} \frac{16k+5}{18^{2k}} \binom{2k}{k} g_k(-20) = \frac{189}{25\pi}. \quad (3.20)$$

We also have

$$\sum_{n=0}^{\infty} \frac{21n+1}{64^n} \sum_{k=0}^n \binom{n}{k} \binom{2k}{n} \binom{2k}{k} \binom{2n-2k}{n-k} 3^{2k-n} = \frac{64}{\pi}. \quad (3.21)$$

Remark. (a) I would like to offer \$520 (520 US dollars) for the person who could give the first correct proof of (3.18) in 2012 because May 20 is the day for Nanjing University.

(b) For $n = 0, 1, 2, \dots$ define

$$f_n(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} x^k = \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{k}{n-k} x^k.$$

Then

$$f_n(1) = \sum_{k=0}^n \binom{n}{k}^3, \quad f_n^+(x) = x^{-n} f_n(x^2) \quad \text{and} \quad f_n^-(x) = x^{-n} f_n(-x^2).$$

By [S11d], $\sum_{k=0}^n \binom{n}{k} (-1)^k ((-1)^k f_k(x)) = g_n(x)$. Thus, by the technique in Section 4, each of (3.11)-(3.19) has an equivalent form in term of $g_k(x)$. Namely, (3.11)-(3.19) are equivalent to the following (3.11')-(3.19') respectively.

$$\sum_{k=0}^{\infty} \frac{720k + 113}{38^{2k}} \binom{2k}{k} g_k(36) = \frac{2527\sqrt{15}}{12\pi}, \quad (3.11')$$

$$\sum_{k=0}^{\infty} \frac{17k + 1}{4050^k} \binom{2k}{k} g_k(196) = \frac{15525}{98\sqrt{7}\pi}, \quad (3.12')$$

$$\sum_{k=0}^{\infty} \frac{3920k + 541}{198^{2k}} \binom{2k}{k} g_k(196) = \frac{42471}{8\sqrt{7}\pi}, \quad (3.13')$$

$$\sum_{k=0}^{\infty} \frac{2352k + 241}{110^{2k}} \binom{2k}{k} g_k(441) = \frac{39325}{6\sqrt{3}\pi}, \quad (3.14')$$

$$\sum_{k=0}^{\infty} \frac{18139680k + 1983409}{1298^{2k}} \binom{2k}{k} g_k(1296) = \frac{109091059}{12\sqrt{2}\pi}, \quad (3.15')$$

$$\sum_{k=0}^{\infty} \frac{944607040k + 86734691}{5778^{2k}} \binom{2k}{k} g_k(5776) = \frac{1071111195\sqrt{95}}{38\pi}, \quad (3.16')$$

$$\sum_{k=0}^{\infty} \frac{35819000k + 5488597}{(-5177196)^k} \binom{2k}{k} g_k(30625) = \frac{3315222\sqrt{19}}{\pi}, \quad (3.17')$$

$$\sum_{k=0}^{\infty} \frac{5440k + 1201}{62^{2k}} \binom{2k}{k} g_k(-64) = \frac{12493\sqrt{15}}{18\pi}, \quad (3.18')$$

$$\sum_{k=0}^{\infty} \frac{1505520k + 220333}{322^{2k}} \binom{2k}{k} g_k(-324) = \frac{1684865\sqrt{5}}{6\pi}. \quad (3.19')$$

Note that [CTYZ] contains some series for $1/\pi$ involving $f_k = f_k(1)$ or $g_k = g_k(1)$.

(c) Observe that

$$\sum_{k=0}^n \binom{n}{k} \binom{2k}{n} \binom{2k}{k} \binom{2n-2k}{n-k} (-1)^{n-k} = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k},$$

which can be proved by obtaining the same recurrence relation for both sides via the Zeilberger algorithm.

Conjecture 4 (Discovered during April 23-25 and May 7-16, 2011).

(i) *We have*

$$\sum_{n=0}^{\infty} \frac{8n+1}{9^n} \sum_{k=0}^n \binom{-1/3}{k}^2 \binom{-2/3}{n-k}^2 = \frac{3\sqrt{3}}{\pi}, \quad (4.1)$$

$$\sum_{n=0}^{\infty} \frac{(2n-1)(-3)^n}{16^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/3}{k} \binom{-2/3}{n-k} = \frac{16}{\sqrt{3}\pi}, \quad (4.2)$$

$$\sum_{n=0}^{\infty} \frac{10n+3}{16^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/3}{k} \binom{-2/3}{n-k} = \frac{16\sqrt{3}}{5\pi}, \quad (4.3)$$

$$\sum_{n=0}^{\infty} \frac{8n+1}{(-20)^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/3}{k} \binom{-2/3}{n-k} = \frac{4\sqrt{3}}{\pi}, \quad (4.4)$$

$$\sum_{n=0}^{\infty} \frac{168n+29}{108^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/3}{k} \binom{-2/3}{n-k} = \frac{324\sqrt{3}}{7\pi}, \quad (4.5)$$

$$\sum_{n=0}^{\infty} \frac{162n+23}{(-112)^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/3}{k} \binom{-2/3}{n-k} = \frac{48\sqrt{3}}{\pi}. \quad (4.6)$$

Also,

$$\sum_{n=0}^{\infty} \frac{(n-2)(-2)^n}{9^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/4}{k} \binom{-3/4}{n-k} = \frac{6\sqrt{3}}{\pi}, \quad (4.7)$$

$$\sum_{n=0}^{\infty} \frac{16n+5}{12^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/4}{k} \binom{-3/4}{n-k} = \frac{8}{\pi}, \quad (4.8)$$

$$\sum_{n=0}^{\infty} \frac{12n+1}{(-16)^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/4}{k} \binom{-3/4}{n-k} = \frac{32}{3\pi}, \quad (4.9)$$

and

$$\sum_{n=0}^{\infty} \frac{(81n+32)8^n}{49^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/4}{k} \binom{-3/4}{n-k} = \frac{14\sqrt{7}}{\pi}, \quad (4.10)$$

$$\sum_{n=0}^{\infty} \frac{n(-8)^n}{81^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/4}{k} \binom{-3/4}{n-k} = \frac{5}{4\pi} ? \quad (4.11)$$

$$\sum_{n=0}^{\infty} \frac{324n+43}{320^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/4}{k} \binom{-3/4}{n-k} = \frac{128}{\pi}, \quad (4.12)$$

$$\sum_{n=0}^{\infty} \frac{320n+39}{(-324)^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{-1/4}{k} \binom{-3/4}{n-k} = \frac{648}{5\pi}. \quad (4.13)$$

(ii) *We have*

$$\sum_{n=0}^{\infty} \frac{3n-1}{2^n} \sum_{k=0}^n \binom{-1/3}{k} \binom{-2/3}{n-k} \binom{-1/6}{k} \binom{-5/6}{n-k} = \frac{3\sqrt{6}}{2\pi}. \quad (4.14)$$

If we set

$$\begin{aligned} a_n &= \sum_{k=0}^n (-1)^k \binom{-1/3}{k}^2 \binom{-2/3}{n-k} \\ &= \sum_{k=0}^n (-1)^k \binom{-2/3}{k}^2 \binom{-1/3}{n-k} \\ &= \frac{(-4)^n}{\binom{2n}{n}} \sum_{k=0}^n \binom{-2/3}{k} \binom{-1/3}{n-k} \binom{-1/6}{k} \binom{-5/6}{n-k}, \end{aligned}$$

then

$$\sum_{n=0}^{\infty} \frac{3n-2}{(-5)^n} \binom{2n}{n} a_n = \frac{3\sqrt{15}}{\pi}, \quad (4.15)$$

$$\sum_{n=0}^{\infty} \frac{32n+1}{(-100)^n} 9^n \binom{2n}{n} a_n = \frac{50}{\sqrt{3}\pi}, \quad (4.16)$$

$$\sum_{n=0}^{\infty} \frac{81n+13}{50^n} \binom{2n}{n} a_n = \frac{75\sqrt{3}}{4\pi}, \quad (4.17)$$

$$\sum_{n=0}^{\infty} \frac{96n+11}{(-68)^n} \binom{2n}{n} a_n = \frac{6\sqrt{51}}{\pi}, \quad (4.18)$$

$$\sum_{n=0}^{\infty} \frac{15n+2}{121^n} \binom{2n}{n} a_n = \frac{363\sqrt{15}}{250\pi}, \quad (4.19)$$

$$\sum_{n=0}^{\infty} \frac{160n+17}{(-324)^n} \binom{2n}{n} a_n = \frac{16}{\sqrt{3}\pi}, \quad (4.20)$$

$$\sum_{n=0}^{\infty} \frac{6144n+527}{(-4100)^n} \binom{2n}{n} a_n = \frac{150\sqrt{123}}{\pi}, \quad (4.21)$$

$$\sum_{n=0}^{\infty} \frac{1500000n+87659}{(-1000004)^n} \binom{2n}{n} a_n = \frac{16854\sqrt{267}}{\pi}. \quad (4.22)$$

(iii) For $n = 0, 1, 2, \dots$ set

$$\begin{aligned} b_n &= \sum_{k=0}^n (-1)^k \binom{-1/4}{k}^2 \binom{-3/4}{n-k} \\ &= \sum_{k=0}^n (-1)^k \binom{-3/4}{k}^2 \binom{-1/4}{n-k} \\ &= \frac{(-4)^n}{\binom{2n}{n}} \sum_{k=0}^n \binom{-1/8}{k} \binom{-5/8}{k} \binom{-3/8}{n-k} \binom{-7/8}{n-k}. \end{aligned}$$

Then

$$\sum_{n=0}^{\infty} \frac{16n+1}{(-20)^n} \binom{2n}{n} b_n = \frac{4\sqrt{5}}{\pi}, \quad (4.23)$$

$$\sum_{n=0}^{\infty} \frac{(3n-1)4^n}{(-25)^n} \binom{2n}{n} b_n = \frac{25}{3\sqrt{3}\pi}, \quad (4.24)$$

$$\sum_{n=0}^{\infty} \frac{6n+1}{32^n} \binom{2n}{n} b_n = \frac{8\sqrt{6}}{9\pi}, \quad (4.25)$$

$$\sum_{n=0}^{\infty} \frac{81n+23}{49^n} 8^n \binom{2n}{n} b_n = \frac{49}{2\pi}, \quad (4.26)$$

$$\sum_{n=0}^{\infty} \frac{192n+19}{(-196)^n} \binom{2n}{n} b_n = \frac{196}{3\pi}, \quad (4.27)$$

$$\sum_{n=0}^{\infty} \frac{162n+17}{320^n} \binom{2n}{n} b_n = \frac{16\sqrt{10}}{\pi}, \quad (4.28)$$

$$\sum_{n=0}^{\infty} \frac{1296n+113}{(-1300)^n} \binom{2n}{n} b_n = \frac{100\sqrt{13}}{\pi}, \quad (4.29)$$

$$\sum_{n=0}^{\infty} \frac{4802n+361}{9600^n} \binom{2n}{n} b_n = \frac{800\sqrt{2}}{\pi}, \quad (4.30)$$

$$\sum_{n=0}^{\infty} \frac{162n+11}{39200^n} \binom{2n}{n} b_n = \frac{19600}{121\sqrt{22}\pi}. \quad (4.31)$$

(iv) For $n = 0, 1, 2, \dots$ set

$$\begin{aligned} c_n &:= \sum_{k=0}^n (-1)^k \binom{-1/6}{k}^2 \binom{-5/6}{n-k} \\ &= \sum_{k=0}^n (-1)^k \binom{-5/6}{k}^2 \binom{-1/6}{n-k} \\ &= \frac{(-4)^n}{\binom{2n}{n}} \sum_{k=0}^n \binom{-1/12}{k} \binom{-7/12}{k} \binom{-5/12}{n-k} \binom{-11/12}{n-k}. \end{aligned}$$

Then we have

$$\sum_{n=0}^{\infty} \frac{125n+13}{121^n} \binom{2n}{n} c_n = \frac{121}{2\sqrt{3}\pi}, \quad (4.32)$$

$$\sum_{n=0}^{\infty} \frac{(125n-8)16^n}{(-189)^n} \binom{2n}{n} c_n = \frac{27\sqrt{7}}{\pi}, \quad (4.33)$$

$$\sum_{n=0}^{\infty} \frac{(125n+24)27^n}{392^n} \binom{2n}{n} c_n = \frac{49}{\sqrt{2}\pi}, \quad (4.34)$$

$$\sum_{n=0}^{\infty} \frac{512n+37}{(-2052)^n} \binom{2n}{n} c_n = \frac{27\sqrt{19}}{\pi}. \quad (4.35)$$

$$\sum_{n=0}^{\infty} \frac{(512n+39)27^n}{(-2156)^n} \binom{2n}{n} c_n = \frac{49\sqrt{11}}{\pi}, \quad (4.36)$$

$$\sum_{n=0}^{\infty} \frac{(1331n+109)2^n}{1323^n} \binom{2n}{n} c_n = \frac{1323}{4\pi}. \quad (4.37)$$

Remark. (i) I [S11c] proved the following three identities:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{n}{4^n} \sum_{k=0}^n \binom{-1/4}{k}^2 \binom{-3/4}{n-k}^2 &= \frac{4\sqrt{3}}{9\pi}, \\ \sum_{n=0}^{\infty} \frac{9n+2}{(-8)^n} \sum_{k=0}^n \binom{-1/4}{k}^2 \binom{-3/4}{n-k}^2 &= \frac{4}{\pi}, \\ \sum_{n=0}^{\infty} \frac{9n+1}{64^n} \sum_{k=0}^n \binom{-1/4}{k}^2 \binom{-3/4}{n-k}^2 &= \frac{64}{7\sqrt{7}\pi}. \end{aligned}$$

On May 15, 2011 I observed that if $x+y+1=0$ then

$$\sum_{k=0}^n (-1)^k \binom{x}{k}^2 \binom{y}{n-k} = \sum_{k=0}^n (-1)^k \binom{y}{k}^2 \binom{x}{n-k}$$

which can be easily proved since both sides satisfy the same recurrence relation by the Zeilberger algorithm. Also,

$$\begin{aligned} \sum_{k=0}^n \binom{-1/3}{k} \binom{-2/3}{n-k} \binom{-1/6}{k} \binom{-5/6}{n-k} &= \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \frac{a_k}{4^k}, \\ \sum_{k=0}^n \binom{-1/8}{k} \binom{-3/8}{n-k} \binom{-5/8}{k} \binom{-7/8}{n-k} &= \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \frac{b_k}{4^k}, \\ \sum_{k=0}^n \binom{-1/12}{k} \binom{-5/12}{n-k} \binom{-7/12}{k} \binom{-11/12}{n-k} &= \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \frac{c_k}{4^k}. \end{aligned}$$

Thus, each of (4.15)-(4.37) has an equivalent form since

$$\sum_{n=0}^{\infty} \frac{bn+c}{m^n} \sum_{k=0}^n \binom{n}{k} (-1)^k f(k) = \frac{m}{(m-1)^2} \sum_{k=0}^{\infty} \frac{bmk+b+(m-1)c}{(1-m)^k} f(k)$$

if both series in the equality converge absolutely. For example, (4.22) holds if and only if

$$\sum_{n=0}^{\infty} \frac{16854n+985}{(-250000)^n} \sum_{k=0}^n \binom{-1/3}{k} \binom{-2/3}{n-k} \binom{-1/6}{k} \binom{-5/6}{n-k} = \frac{4500000}{89\sqrt{267}\pi}.$$

(ii) (4.1) and (4.14) appeared as conjectures in [S11b]. In December 2011 G. Almkvist and A. Aycok released the preprint [AA] in which they proved all the conjectured formulas in Conj. 4 except (4.14), with the right-hand side of (4.11) corrected as $162/(49\sqrt{7}\pi)$.

Conjecture 5 ((5.1)-(5.9) and (5.10) were discovered during June 16-17, 2011 and on Jan. 18, 2012 respectively). *Define*

$$s_n(x) := \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} x^{-(n+k)} \quad \text{for } n = 0, 1, 2, \dots$$

Then

$$\sum_{k=0}^{\infty} (7k+2) \binom{2k}{k} s_k(-9) = \frac{9\sqrt{3}}{5\pi}, \quad (5.1)$$

$$\sum_{k=0}^{\infty} (9k+2) \binom{2k}{k} s_k(-20) = \frac{4}{\pi}, \quad (5.2)$$

$$\sum_{k=0}^{\infty} (95k+13) \binom{2k}{k} s_k(36) = \frac{18\sqrt{15}}{\pi}, \quad (5.3)$$

$$\sum_{k=0}^{\infty} (310k+49) \binom{2k}{k} s_k(-64) = \frac{32\sqrt{15}}{\pi}, \quad (5.4)$$

$$\sum_{k=0}^{\infty} (495k+53) \binom{2k}{k} s_k(196) = \frac{70\sqrt{7}}{\pi}, \quad (5.5)$$

$$\sum_{k=0}^{\infty} (13685k+1474) \binom{2k}{k} s_k(-324) = \frac{1944\sqrt{5}}{\pi}, \quad (5.6)$$

$$\sum_{k=0}^{\infty} (3245k+268) \binom{2k}{k} s_k(1296) = \frac{1215}{\sqrt{2}\pi}, \quad (5.7)$$

$$\sum_{k=0}^{\infty} (6420k+443) \binom{2k}{k} s_k(5776) = \frac{1292\sqrt{95}}{9\pi}. \quad (5.8)$$

Also,

$$\sum_{n=0}^{\infty} \frac{357n+103}{2160^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k} = \frac{90}{\pi}, \quad (5.9)$$

$$\sum_{n=0}^{\infty} \frac{n}{3645^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} 486^{n-k} = \frac{10}{3\pi}. \quad (5.10)$$

Remark. I would like to offer \$90 for the first rigorous proof of (5.9), and \$1050 for the first complete proof of my following related conjecture: For any prime $p > 5$, we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{357n+103}{2160^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k} \\ & \equiv p \left(\frac{-1}{p} \right) \left(54 + 49 \left(\frac{p}{15} \right) \right) \pmod{p^2}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{2160^n} \sum_{k=0}^n \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p} \right) = \left(\frac{p}{3} \right) = \left(\frac{p}{5} \right) = \left(\frac{p}{7} \right) = 1, p = x^2 + 105y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p} \right) = \left(\frac{p}{7} \right) = 1, \left(\frac{p}{3} \right) = \left(\frac{p}{5} \right) = -1, 2p = x^2 + 105y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p} \right) = \left(\frac{p}{3} \right) = \left(\frac{p}{5} \right) = \left(\frac{p}{7} \right) = -1, p = 3x^2 + 35y^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } \left(\frac{-1}{p} \right) = \left(\frac{p}{7} \right) = -1, \left(\frac{p}{3} \right) = \left(\frac{p}{5} \right) = 1, 2p = 3x^2 + 35y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p} \right) = \left(\frac{p}{5} \right) = 1, \left(\frac{p}{3} \right) = \left(\frac{p}{7} \right) = -1, p = 5x^2 + 21y^2, \\ 10x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p} \right) = \left(\frac{p}{3} \right) = 1, \left(\frac{p}{5} \right) = \left(\frac{p}{7} \right) = -1, 2p = 5x^2 + 21y^2, \\ 28x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p} \right) = \left(\frac{p}{5} \right) = -1, \left(\frac{p}{3} \right) = \left(\frac{p}{7} \right) = 1, p = 7x^2 + 15y^2, \\ 14x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{-1}{p} \right) = \left(\frac{p}{3} \right) = -1, \left(\frac{p}{5} \right) = \left(\frac{p}{7} \right) = 1, 2p = 7x^2 + 15y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-105}{p} \right) = -1. \end{cases} \end{aligned}$$

(Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number eight.) In fact, for all series for $1/\pi$ that I found, I had such conjectures on congruences. See [S11b] for my philosophy about series for $1/\pi$.

Conjecture 6. (i) ([S11b]) *We have*

$$\sum_{n=0}^{\infty} \frac{114n+31}{26^{2n}} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} (-27)^k = \frac{338\sqrt{3}}{11\pi}, \quad (6.1)$$

$$\sum_{n=0}^{\infty} \frac{930n+143}{28^{2n}} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} 27^k = \frac{980\sqrt{3}}{\pi}. \quad (6.2)$$

(ii) (Discovered on Jan. 12, 2012) *Set*

$$P_n(x) = \sum_{k=0}^n \binom{2k}{k}^2 \binom{k}{n-k} x^{n-k} \quad \text{for } n = 0, 1, 2, \dots$$

Then we have

$$\sum_{k=0}^{\infty} \frac{14k+3}{8^{2k}} \binom{2k}{k} P_k(-7) = \frac{16\sqrt{7}}{3\pi}, \quad (6.3)$$

$$\sum_{k=0}^{\infty} \frac{255k+56}{13^{2k}} \binom{2k}{k} P_k(14) = \frac{2028}{7\pi}, \quad (6.4)$$

$$\sum_{k=0}^{\infty} \frac{308k+59}{20^{2k}} \binom{2k}{k} P_k(21) = \frac{250\sqrt{7}}{3\pi}, \quad (6.5)$$

$$\sum_{k=0}^{\infty} \frac{1932k+295}{44^{2k}} \binom{2k}{k} P_k(45) = \frac{363\sqrt{7}}{\pi}, \quad (6.6)$$

$$\sum_{k=0}^{\infty} \frac{890358k+97579}{176^{2k}} \binom{2k}{k} P_k(-175) = \frac{116160\sqrt{7}}{\pi}. \quad (6.7)$$

(iii) (Discovered on Jan. 14, 2012) *Define*

$$s_n = \sum_{k=0}^n 5^k \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2 / \binom{n}{k} \quad \text{for } n = 0, 1, 2, \dots$$

Then we have

$$\sum_{k=0}^{\infty} \frac{28k+5}{576^k} \binom{2k}{k} s_k = \frac{9}{\pi} (2 + \sqrt{2}). \quad (6.8)$$

Remark. It is known that $\binom{n}{k} \mid \binom{2k}{k} \binom{2(n-k)}{n-k}$ for all $k = 0, \dots, n$. Recall that the Catalan-Larcombe-French numbers P_0, P_1, \dots are given by

$$P_n = \sum_{k=0}^n \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} = 2^n P_n(-4) = 2^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}^2 4^{n-2k}$$

and these numbers satisfy the recurrence relation

$$(k+1)^2 P_{k+1} = (24k(k+1) + 8) P_k - 128k^2 P_{k-1} \quad (k = 1, 2, 3, \dots).$$

Note that

$$\sum_{k=0}^n (-1)^k \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} = \begin{cases} 4^n \binom{n}{n/2}^2 & \text{if } 2 \mid n, \\ 0 & \text{if } 2 \nmid n. \end{cases}$$

The sequence $\{s_n\}_{n \geq 0}$ can also be defined by $s_0 = 1$, $s_1 = 24$, $s_2 = 976$ and the recurrence relation

$$51200(n+1)^2(n+3)s_n - 1920(4n^3 + 24n^2 + 46n + 29)s_{n+1} \\ + 8(n+2)(41n^2 + 205n + 255)s_{n+2} - 3(n+2)(n+3)^2s_{n+3} = 0.$$

A sequence of polynomials $\{P_n(q)\}_{n \geq 0}$ with integer coefficients is said to be *q-logconvex* if for each $n = 1, 2, 3, \dots$ all the coefficients of the polynomial $P_{n-1}(q)P_{n+1}(q) - P_n(q)^2 \in \mathbb{Z}[q]$ are nonnegative. In view of Conjectures 2 and 3, on May 7, 2011 I conjectured that $\{P_n(q)\}_{n \geq 0}$ is *q-logconvex* if $P_n(q)$ has one of the following forms:

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} q^k, \quad \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} q^k, \\ \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} q^k, \quad \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k}^2 \binom{2(n-k)}{n-k} q^k.$$

2. SERIES FOR $1/\pi$ INVOLVING GENERALIZED CENTRAL TRINOMIAL COEFFICIENTS

For $b, c \in \mathbb{Z}$, the *generalized central trinomial coefficient* $T_n(b, c)$ denotes the coefficient of x^n in the expansion of $(x^2 + bx + c)^n$. It is easy to see that

$$T_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} b^{n-2k} c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} b^{n-2k} c^k.$$

An efficient way to compute $T_n(b, c)$ is to use the initial values

$$T_0(b, c) = 1, \quad T_1(b, c) = b,$$

and the recursion

$$(n+1)T_{n+1}(b, c) = (2n+1)bT_n(b, c) - n(b^2 - 4c)T_{n-1}(b, c) \quad (n = 1, 2, \dots).$$

In view of the Laplace-Heine asymptotic formula for Legendre polynomials, I [S11a] noted that for any positive reals b and c we have

$$T_n(b, c) \sim \frac{(b + 2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}$$

as $n \rightarrow +\infty$.

In Jan.-Feb. 2011, I introduced a number of series for $1/\pi$ of the following new types with a, b, c, d, m integers and $mbcd(b^2 - 4c)$ nonzero.

- Type I. $\sum_{k=0}^{\infty} (a + dk) \binom{2k}{k}^2 T_k(b, c) / m^k$.
 Type II. $\sum_{k=0}^{\infty} (a + dk) \binom{2k}{k} \binom{3k}{k} T_k(b, c) / m^k$.
 Type III. $\sum_{k=0}^{\infty} (a + dk) \binom{4k}{2k} \binom{2k}{k} T_k(b, c) / m^k$.
 Type IV. $\sum_{k=0}^{\infty} (a + dk) \binom{2k}{k}^2 T_{2k}(b, c) / m^k$.
 Type V. $\sum_{k=0}^{\infty} (a + dk) \binom{2k}{k} \binom{3k}{k} T_{3k}(b, c) / m^k$.

During October 1-3, 2011, I introduced two new kinds of series for $1/\pi$:

- Type VI. $\sum_{k=0}^{\infty} (a + dk) T_k(b, c)^3 / m^k$,
 Type VII. $\sum_{k=0}^{\infty} (a + dk) \binom{2k}{k} T_k(b, c)^2 / m^k$,

where a, b, c, d, m are integers and $mbcd(b^2 - 4c)$ is nonzero.

Recall that a series $\sum_{k=0}^{\infty} a_k$ is said to converge at a geometric rate with ratio r if $\lim_{k \rightarrow +\infty} a_{k+1}/a_k = r \in (0, 1)$.

Conjecture I (Z. W. Sun [S11a]). *We have the following identities:*

$$\sum_{k=0}^{\infty} \frac{30k+7}{(-256)^k} \binom{2k}{k}^2 T_k(1, 16) = \frac{24}{\pi}, \quad (\text{I1})$$

$$\sum_{k=0}^{\infty} \frac{30k+7}{(-1024)^k} \binom{2k}{k}^2 T_k(34, 1) = \frac{12}{\pi}, \quad (\text{I2})$$

$$\sum_{k=0}^{\infty} \frac{30k-1}{4096^k} \binom{2k}{k}^2 T_k(194, 1) = \frac{80}{\pi}, \quad (\text{I3})$$

$$\sum_{k=0}^{\infty} \frac{42k+5}{4096^k} \binom{2k}{k}^2 T_k(62, 1) = \frac{16\sqrt{3}}{\pi}. \quad (\text{I4})$$

Remark. The series (I1)-(I4) converge at geometric rates with ratios $-9/16$, $-9/16$, $49/64$, $1/4$ respectively.

Conjecture II (Z. W. Sun [S11a]). *We have*

$$\sum_{k=0}^{\infty} \frac{15k+2}{972^k} \binom{2k}{k} \binom{3k}{k} T_k(18, 6) = \frac{45\sqrt{3}}{4\pi}, \quad (\text{II1})$$

$$\sum_{k=0}^{\infty} \frac{91k+12}{10^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(10, 1) = \frac{75\sqrt{3}}{2\pi}, \quad (\text{II2})$$

$$\sum_{k=0}^{\infty} \frac{15k-4}{18^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(198, 1) = \frac{135\sqrt{3}}{2\pi}, \quad (\text{II3})$$

$$\sum_{k=0}^{\infty} \frac{42k-41}{30^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(970, 1) = \frac{525\sqrt{3}}{\pi}, \quad (\text{II4})$$

$$\sum_{k=0}^{\infty} \frac{18k+1}{30^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(730, 729) = \frac{25\sqrt{3}}{\pi}, \quad (\text{II5})$$

$$\sum_{k=0}^{\infty} \frac{6930k + 559}{102^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(102, 1) = \frac{1445\sqrt{6}}{2\pi}, \quad (\text{II6})$$

$$\sum_{k=0}^{\infty} \frac{222105k + 15724}{198^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(198, 1) = \frac{114345\sqrt{3}}{4\pi}, \quad (\text{II7})$$

$$\sum_{k=0}^{\infty} \frac{390k - 3967}{102^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(39202, 1) = \frac{56355\sqrt{3}}{\pi}, \quad (\text{II8})$$

$$\sum_{k=0}^{\infty} \frac{210k - 7157}{198^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(287298, 1) = \frac{114345\sqrt{3}}{\pi}, \quad (\text{II9})$$

and

$$\sum_{k=0}^{\infty} \frac{45k + 7}{24^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(26, 729) = \frac{8}{3\pi} (3\sqrt{3} + \sqrt{15}), \quad (\text{II10})$$

$$\sum_{k=0}^{\infty} \frac{9k + 2}{(-5400)^k} \binom{2k}{k} \binom{3k}{k} T_k(70, 3645) = \frac{15\sqrt{3} + \sqrt{15}}{6\pi}, \quad (\text{II11})$$

$$\sum_{k=0}^{\infty} \frac{63k + 11}{(-13500)^k} \binom{2k}{k} \binom{3k}{k} T_k(40, 1458) = \frac{25}{12\pi} (3\sqrt{3} + 4\sqrt{6}), \quad (\text{II12})$$

Remark. The series (II1)-(II12) converge at geometric rates with ratios

$$\begin{aligned} & \frac{9 + \sqrt{6}}{18}, \frac{81}{250}, \frac{25}{27}, \frac{243}{250}, \frac{98}{125}, \frac{13}{4913}, \frac{25}{35937}, \\ & \frac{9801}{9826}, \frac{71825}{71874}, \frac{5}{32}, -\frac{35 + 27\sqrt{5}}{100}, -\frac{20 + 27\sqrt{2}}{250} \end{aligned}$$

respectively.

Conjecture III (Z. W. Sun [S11a]). *We have the following formulae:*

$$\sum_{k=0}^{\infty} \frac{85k + 2}{66^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(52, 1) = \frac{33\sqrt{33}}{\pi}, \quad (\text{III1})$$

$$\sum_{k=0}^{\infty} \frac{28k + 5}{(-96^2)^k} \binom{4k}{2k} \binom{2k}{k} T_k(110, 1) = \frac{3\sqrt{6}}{\pi}, \quad (\text{III2})$$

$$\sum_{k=0}^{\infty} \frac{40k + 3}{112^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(98, 1) = \frac{70\sqrt{21}}{9\pi}, \quad (\text{III3})$$

$$\sum_{k=0}^{\infty} \frac{80k + 9}{264^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(257, 256) = \frac{11\sqrt{66}}{2\pi}, \quad (\text{III4})$$

$$\sum_{k=0}^{\infty} \frac{80k + 13}{(-168^2)^k} \binom{4k}{2k} \binom{2k}{k} T_k(7, 4096) = \frac{14\sqrt{210} + 21\sqrt{42}}{8\pi}, \quad (\text{III5})$$

$$\sum_{k=0}^{\infty} \frac{760k + 71}{336^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(322, 1) = \frac{126\sqrt{7}}{\pi}, \quad (\text{III6})$$

$$\sum_{k=0}^{\infty} \frac{10k - 1}{336^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(1442, 1) = \frac{7\sqrt{210}}{4\pi}, \quad (\text{III7})$$

$$\sum_{k=0}^{\infty} \frac{770k + 69}{912^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(898, 1) = \frac{95\sqrt{114}}{4\pi}, \quad (\text{III8})$$

$$\sum_{k=0}^{\infty} \frac{280k - 139}{912^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(12098, 1) = \frac{95\sqrt{399}}{\pi}, \quad (\text{III9})$$

and

$$\sum_{k=0}^{\infty} \frac{84370k + 6011}{10416^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(10402, 1) = \frac{3689\sqrt{434}}{4\pi}, \quad (\text{III10})$$

$$\sum_{k=0}^{\infty} \frac{8840k - 50087}{10416^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(1684802, 1) = \frac{7378\sqrt{8463}}{\pi}, \quad (\text{III11})$$

$$\sum_{k=0}^{\infty} \frac{11657240k + 732103}{39216^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(39202, 1) = \frac{80883\sqrt{817}}{\pi}, \quad (\text{III12})$$

$$\sum_{k=0}^{\infty} \frac{3080k - 58871}{39216^{2k}} \binom{4k}{2k} \binom{2k}{k} T_k(23990402, 1) = \frac{17974\sqrt{2451}}{\pi}. \quad (\text{III13})$$

Remark. The series (III1)-(III13) converge at geometric rates with ratios

$$\begin{aligned} & \frac{96}{121}, -\frac{7}{9}, \frac{25}{49}, \frac{289}{1089}, -\frac{15}{49}, \frac{9}{49}, \frac{361}{441}, \\ & \frac{25}{361}, \frac{3025}{3249}, \frac{289}{47089}, \frac{421201}{423801}, \frac{1089}{667489}, \frac{5997601}{6007401} \end{aligned}$$

respectively. I thank Prof. Qing-Hu Hou (at Nankai University) for helping me check (III9) numerically.

Conjecture IV (Z. W. Sun [S11a]). *We have*

$$\sum_{k=0}^{\infty} \frac{26k+5}{(-48^2)^k} \binom{2k}{k}^2 T_{2k}(7, 1) = \frac{48}{5\pi}, \quad (\text{IV1})$$

$$\sum_{k=0}^{\infty} \frac{340k+59}{(-480^2)^k} \binom{2k}{k}^2 T_{2k}(62, 1) = \frac{120}{\pi}, \quad (\text{IV2})$$

$$\sum_{k=0}^{\infty} \frac{13940k+1559}{(-5760^2)^k} \binom{2k}{k}^2 T_{2k}(322, 1) = \frac{4320}{\pi}, \quad (\text{IV3})$$

$$\sum_{k=0}^{\infty} \frac{8k+1}{96^{2k}} \binom{2k}{k}^2 T_{2k}(10, 1) = \frac{10\sqrt{2}}{3\pi}, \quad (\text{IV4})$$

$$\sum_{k=0}^{\infty} \frac{10k+1}{240^{2k}} \binom{2k}{k}^2 T_{2k}(38, 1) = \frac{15\sqrt{6}}{4\pi}, \quad (\text{IV5})$$

$$\sum_{k=0}^{\infty} \frac{14280k+899}{39200^{2k}} \binom{2k}{k}^2 T_{2k}(198, 1) = \frac{1155\sqrt{6}}{\pi}, \quad (\text{IV6})$$

and

$$\sum_{k=0}^{\infty} \frac{120k+13}{320^{2k}} \binom{2k}{k}^2 T_{2k}(18, 1) = \frac{12\sqrt{15}}{\pi}, \quad (\text{IV7})$$

$$\sum_{k=0}^{\infty} \frac{21k+2}{896^{2k}} \binom{2k}{k}^2 T_{2k}(30, 1) = \frac{5\sqrt{7}}{2\pi}, \quad (\text{IV8})$$

$$\sum_{k=0}^{\infty} \frac{56k+3}{24^{4k}} \binom{2k}{k}^2 T_{2k}(110, 1) = \frac{30\sqrt{7}}{\pi}, \quad (\text{IV9})$$

$$\sum_{k=0}^{\infty} \frac{56k+5}{48^{4k}} \binom{2k}{k}^2 T_{2k}(322, 1) = \frac{72\sqrt{7}}{5\pi}, \quad (\text{IV10})$$

$$\sum_{k=0}^{\infty} \frac{10k+1}{2800^{2k}} \binom{2k}{k}^2 T_{2k}(198, 1) = \frac{25\sqrt{14}}{24\pi}, \quad (\text{IV11})$$

$$\sum_{k=0}^{\infty} \frac{195k+14}{10400^{2k}} \binom{2k}{k}^2 T_{2k}(102, 1) = \frac{85\sqrt{39}}{12\pi}, \quad (\text{IV12})$$

and

$$\sum_{k=0}^{\infty} \frac{3230k + 263}{46800^{2k}} \binom{2k}{k}^2 T_{2k}(1298, 1) = \frac{675\sqrt{26}}{4\pi}, \quad (\text{IV13})$$

$$\sum_{k=0}^{\infty} \frac{520k - 111}{5616^{2k}} \binom{2k}{k}^2 T_{2k}(1298, 1) = \frac{1326\sqrt{3}}{\pi}, \quad (\text{IV14})$$

$$\sum_{k=0}^{\infty} \frac{280k - 149}{20400^{2k}} \binom{2k}{k}^2 T_{2k}(4898, 1) = \frac{330\sqrt{51}}{\pi}, \quad (\text{IV15})$$

$$\sum_{k=0}^{\infty} \frac{78k - 1}{28880^{2k}} \binom{2k}{k}^2 T_{2k}(5778, 1) = \frac{741\sqrt{10}}{20\pi}, \quad (\text{IV16})$$

$$\sum_{k=0}^{\infty} \frac{57720k + 3967}{439280^{2k}} \binom{2k}{k}^2 T_{2k}(5778, 1) = \frac{2890\sqrt{19}}{\pi}, \quad (\text{IV17})$$

$$\sum_{k=0}^{\infty} \frac{1615k - 314}{243360^{2k}} \binom{2k}{k}^2 T_{2k}(54758, 1) = \frac{1989\sqrt{95}}{4\pi}. \quad (\text{IV18})$$

Remark. The series (IV1)-(IV18) converge at geometric rates with ratios

$$\begin{aligned} & -\frac{9}{16}, -\frac{65}{225}, -\frac{81}{1600}, \frac{1}{4}, \frac{4}{9}, \frac{1}{2401}, \frac{1}{16}, \frac{1}{49}, \frac{49}{81}, \\ & \frac{81}{256}, \frac{4}{49}, \frac{1}{625}, \frac{1}{81}, \frac{625}{729}, \frac{2401}{2601}, \frac{83521}{130321}, \frac{1}{361}, \frac{1874161}{2313441} \end{aligned}$$

respectively. I conjecture that (IV1)-(IV18) have exhausted all identities of the form

$$\sum_{k=0}^{\infty} (a + dk) \frac{\binom{2k}{k}^2 T_{2k}(b, 1)}{m^k} = \frac{C}{\pi}$$

with $a, d, m \in \mathbb{Z}$, $b \in \{1, 3, 4, \dots\}$, $d > 0$, and C^2 positive and rational.

Conjecture V (Z. W. Sun [S11a]). *We have the formula*

$$\sum_{k=0}^{\infty} \frac{1638k + 277}{(-240)^{3k}} \binom{2k}{k} \binom{3k}{k} T_{3k}(62, 1) = \frac{44\sqrt{105}}{\pi}. \quad (\text{V1})$$

Remark. The series (V1) converges at a geometric rate with ratio $-64/125$.

Note that [CWZ1] contains complete proofs of (I2), (I4), (II1), (II11), (III3) and (III5). Also, a detailed proof of (IV4) was given in [WZ]. The most crucial parts of such proofs involve modular equations, so (in my opinion) a complete proof should contain all the details involving modular equation.

Conjecture VI (Z. W. Sun [S11c]). *We have the following formulae:*

$$\sum_{k=0}^{\infty} \frac{66k+17}{(2^{11}3^3)^k} T_k^3(10, 11^2) = \frac{540\sqrt{2}}{11\pi}, \quad (\text{VI1})$$

$$\sum_{k=0}^{\infty} \frac{126k+31}{(-80)^{3k}} T_k^3(22, 21^2) = \frac{880\sqrt{5}}{21\pi}, \quad (\text{VI2})$$

$$\sum_{k=0}^{\infty} \frac{3990k+1147}{(-288)^{3k}} T_k^3(62, 95^2) = \frac{432}{95\pi} (195\sqrt{14} + 94\sqrt{2}). \quad (\text{VI3})$$

Remark. The series (VI1)-(VI3) converge at geometric rates with ratios

$$\frac{16}{27}, \quad -\frac{64}{125}, \quad -\frac{343}{512}$$

respectively. I would like to offer \$300 as the prize for the person (not joint authors) who can provide first rigorous proofs of all the three identities (VI1)-(VI3).

Conjecture VII (Z. W. Sun [S11c]). *We have the following formulae:*

$$\sum_{k=0}^{\infty} \frac{221k+28}{450^k} \binom{2k}{k} T_k^2(6, 2) = \frac{2700}{7\pi}, \quad (\text{VII1})$$

$$\sum_{k=0}^{\infty} \frac{24k+5}{28^{2k}} \binom{2k}{k} T_k^2(4, 9) = \frac{49}{9\pi} (\sqrt{3} + \sqrt{6}), \quad (\text{VII2})$$

$$\sum_{k=0}^{\infty} \frac{560k+71}{22^{2k}} \binom{2k}{k} T_k^2(5, 1) = \frac{605\sqrt{7}}{3\pi}, \quad (\text{VII3})$$

$$\sum_{k=0}^{\infty} \frac{3696k+445}{46^{2k}} \binom{2k}{k} T_k^2(7, 1) = \frac{1587\sqrt{7}}{2\pi}, \quad (\text{VII4})$$

$$\sum_{k=0}^{\infty} \frac{56k+19}{(-108)^k} \binom{2k}{k} T_k^2(3, -3) = \frac{9\sqrt{7}}{\pi}, \quad (\text{VII5})$$

$$\sum_{k=0}^{\infty} \frac{450296k+53323}{(-5177196)^k} \binom{2k}{k} T_k^2(171, -171) = \frac{113535\sqrt{7}}{2\pi}, \quad (\text{VII6})$$

$$\sum_{k=0}^{\infty} \frac{2800512k+435257}{434^{2k}} \binom{2k}{k} T_k^2(73, 576) = \frac{10406669}{2\sqrt{6}\pi}. \quad (\text{VII7})$$

Remark. The series (VII1)-(VII7) converge at geometric rates with ratios

$$\frac{88+48\sqrt{2}}{225}, \quad \frac{25}{49}, \quad \frac{49}{121}, \quad \frac{81}{529}, \quad -\frac{7}{9}, \quad -\frac{175}{7569}, \quad \frac{14641}{47089}.$$

respectively.

3. HISTORICAL NOTES ON THE 58 SERIES IN SECTION 2

I discovered most of those conjectural series for $1/\pi$ in Section 2 during Jan. and Feb. in 2011. Series of type VI and VII were introduced in October 2011. All my conjectural series in Section 2 came from a combination of my philosophy, intuition, inspiration, experience and computation.

In the evening of Jan. 1, 2011 I figured out the asymptotic behavior of $T_n(b, c)$ with b and c positive. (Few days later I learned the Laplace-Heine asymptotic formula for Legendre polynomials and hence knew that my conjectural main term of $T_n(b, c)$ as $n \rightarrow +\infty$ is indeed correct.)

The story of new series for $1/\pi$ began with (I1) which was found in the early morning of Jan. 2, 2011 immediately after I waked up on the bed. On Jan 4 I announced this via a message to **Number Theory Mailing List** as well as the initial version of [S11a] posted to **arXiv**. In the subsequent two weeks I communicated with some experts on π -series and wanted to know whether they could prove my conjectural (I1). On Jan. 20, it seemed clear that series like (I1) could not be easily proved by the current known methods used to establish Ramanujan-type series for $1/\pi$.

Then, I discovered (II1) on Jan. 21 and (III3) on Jan. 29. On Feb. 2 I found (IV1) and (IV4). Then, I discovered (IV2) on Feb. 5. When I waked up in the early morning of Feb. 6, I suddenly realized a (conjectural) criterion for the existence of series for $1/\pi$ of type IV with $c = 1$. Based on this criterion, I found (IV3), (IV5)-(IV10) and (IV12) on Feb. 6, (IV11) on Feb. 7, (IV13) on Feb. 8, (IV14)-(IV16) on Feb. 9, and (IV17) on Feb. 10. On Feb. 14 I discovered (I2)-(I4) and (III4). I found the sophisticated (III5) on Feb. 15. As for series of type IV, I discovered the largest example (IV18) on Feb. 16., and conjectured that the 18 series in Conj. IV have exhausted all those series for $1/\pi$ of type IV with $c = 1$. On Feb. 18 I found (II2), (II5)-(II7), (II10) and (II12).

On Feb. 21 I informed many experts on π -series (including Gert Almkvist) my list of the 34 conjectural series for $1/\pi$ of types I-IV and predicted that there are totally about 40 such series. On Feb. 22 I found (II11) and (II3)-(II4); on the same day, motivated by my conjectural (II2), (II5)-(II7), (II10) and (II12) discovered on Feb 18, G. Almkvist found the following two series of type II that I missed:

$$\sum_{k=0}^{\infty} \frac{42k+5}{18^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(18, 1) = \frac{54\sqrt{3}}{5\pi} \quad (\text{A1})$$

and

$$\sum_{k=0}^{\infty} \frac{66k+7}{30^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(30, 1) = \frac{50\sqrt{2}}{3\pi}. \quad (\text{A2})$$

On Feb. 22, Almkvist also pointed out that my conjectural identity (II2) can be used to compute an arbitrary decimal digit of $\sqrt{3}/\pi$ without computing the earlier digits.

On Feb. 23 I discovered (V1), which is the unique example of series for $1/\pi$ of type V that I can find.

On Feb. 25 and Feb. 26, I found (II8) and (II9) respectively. These two series converge very slowly.

On August 11, I discovered (III6)-(III9) that I missed during Jan.-Feb. (III10)-(III13) were found by me on Sept. 21, 2011. Note that (III13) converges very slow.

On Oct. 1, I discovered (VI2) and (VI3), then I found (VI1) on the next day.

I figured out (VII1)-(VII4), (VII5) and (VII6) on Oct. 3, 4 and 5 respectively. On Oct. 13, 2011 I discovered (VII7).

On Oct. 16 James Wan informed me the preprints [CWZ1] and [WZ] on my conjectural series of types I-V. I admit that these two papers contain complete proofs of (I2), (I4), (II1), (II11), (III3), (III5) and (IV4). Note also that [CWZ2] was motivated by the authors' study of my conjectural (III5).

4. A TECHNIQUE FOR PRODUCING MORE SERIES FOR $1/\pi$

For a sequence a_0, a_1, a_2, \dots of complex numbers, define

$$a_n^* = \sum_{k=0}^n \binom{n}{k} (-1)^k a_k \quad \text{for all } n \in \mathbb{N} = \{0, 1, 2, \dots\}$$

and call $\{a_n^*\}_{n \in \mathbb{N}}$ the *dual sequence* of $\{a_n\}_{n \in \mathbb{N}}$. It is well known that $(a_n^*)^* = a_n$ for all $n \in \mathbb{N}$.

There are many series for $1/\pi$ of the form

$$\sum_{k=0}^{\infty} (bk + c) \frac{\binom{2k}{k} a_k}{m^k} = \frac{C}{\pi}$$

(see Section 2 for many such series). On March 10, 2011, I realized that if $m \neq 0, 4$ then

$$\sum_{n=0}^{\infty} (bmn + 2b + (m-4)c) \frac{\binom{2n}{n} a_n^*}{(4-m)^n} = (m-4) \sqrt{\frac{m-4}{m}} \sum_{k=0}^{\infty} (bk + c) \frac{\binom{2k}{k} a_k}{m^k}. \quad (4.1)$$

This is easy because

$$\begin{aligned}
& \sum_{n=0}^{\infty} (bmn + 2b + (m-4)c) \frac{\binom{2n}{n} a_n^*}{(4-m)^n} \\
&= \sum_{n=0}^{\infty} (bmn + 2b + (m-4)c) \frac{\binom{2n}{n}}{(4-m)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k a_k \\
&= \sum_{k=0}^{\infty} (-1)^k a_k \sum_{n=k}^{\infty} (bmn + 2b + (m-4)c) \frac{\binom{2n}{n} \binom{n}{k}}{(4-m)^n} \\
&= \sum_{k=0}^{\infty} (-1)^k a_k \left((m-4) \sqrt{\frac{m-4}{m}} (bk + c) \frac{\binom{2k}{k}}{(-m)^k} \right) \\
&\quad \text{(by Mathematica or a direct calculation)} \\
&= (m-4) \sqrt{\frac{m-4}{m}} \sum_{k=0}^{\infty} (bk + c) \frac{\binom{2k}{k} a_k}{m^k}.
\end{aligned}$$

Thus, if $m \neq 0, 4$ then

$$\begin{aligned}
& \sum_{k=0}^{\infty} (bk + c) \frac{\binom{2k}{k} a_k}{m^k} = \frac{C}{\pi} \\
& \iff \sum_{k=0}^{\infty} (bmk + 2b + (m-4)c) \frac{\binom{2k}{k} a_k^*}{(4-m)^k} = \frac{(m-4)C}{\pi} \sqrt{\frac{m-4}{m}}.
\end{aligned} \tag{4.2}$$

Example 4.1. Let $a_n = \binom{2n}{n} T_n(1, 16)$ for all $n \in \mathbb{N}$. Then

$$a_n^* = \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} (-1)^k T_k(1, 16) \quad \text{for } n = 0, 1, 2, \dots$$

Thus, by (4.2), the identity (I1) in Section 2 has the following equivalent form:

$$\sum_{k=0}^{\infty} (48k + 11) \frac{\binom{2k}{k} a_k^*}{260^k} = \frac{39\sqrt{65}}{8\pi}.$$

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